TWMS Jour. Pure Appl. Math., V.2, N.2, 2011, pp.271-278

GROUND SUBGROUPS*

U.A. ROZIKOV[1]

Abstract. In this article we give a concept of ground subgroup for finite and countable groups. By our definition such a subgroup of a group depends on a given subset of the group and on a given partition of the subset. For finite and free groups we describe some sets of ground subgroups. We apply the ground subgroups to describe ground states of a model of statistical mechanics.

Keywords: group, subgroup, ground subgroup, ground state, configuration, Hamiltonian.

AMS Subject Classification: 20E07, 20E32.

1. Introduction

There are several thousand papers and books devoted to the theory of groups. But still there are unsolved problems, most of which arise in solving of problems of natural sciences as physics, biology etc. In particular, if configuration of a physical system is located on a lattice (in our case on the graph of a group) then the configuration can be considered as a function defined on the lattice. Usually, more important configurations (functions) are periodic ones. It is well-known that if the lattice has a group representation then periodicity of a function can be defined by a given subgroup of the representation. More precisely, if a subgroup, say H, is given, then one can define H-periodic function as a function, which has a constant value (depending only on the coset) on each (right or left) coset of H. So the periodicity is related to a special partition of the group (that presents the lattice on which our physical system is located). There are many works devoted to several kind of partitions of groups (lattices) (see e.g. [1],[2], [4]-[9], [12]).

In this paper we study more general problem: given a subset A of a finite or countable group G, and given a partition of A, is there any subgroup H of G such that cosets of which divide the set A exactly as the given partition? This problem arises if, for example, one want to study the periodic ground configurations (states) of a physical system with the Hamiltonian (energy) which is a sum of interaction functions I_A defined on A (see e.g. [3], [10], [11]). Note that a Hamiltonian is a function of configurations. A *ground state* of a Hamiltonian is a (periodic) configuration which minimizes the Hamiltonian. Thus subgroups which we want to describe are related to the ground states of a Hamiltonian, so we call them *ground subgroups.*

The paper is organized as follows. In section 2 we give all necessary definitions and formulation of the problem. In sections 3-5 we describe ground subgroups of finite and free groups. A ground subgroup depends on a given subset, say A and on a given partition of the subset A. For finite groups we show that one can choose A and its partition such that there does not exist any ground subgroup corresponding to this partition of the subset. Section 6 is devoted to an application

* Supported by TWAS Research Grant: 09-009 RG/Maths/As-I; UNESCO FR: 3240230333.

[1] Institute of Mathematics and Information Technologies, Tashkent, Uzbekistan,
e-mail: rozikovu@yandex.ru

Manuscript received January 2011.

of results to description of ground states of a Hamiltonian defined on a Cayley tree. In the last section we discuss the results and give some open problems.

2. Definitions and statement of the problem

Let G be a finite or countable group and H be a subgroup of G. For $x \in G$ denote $Hx = = \{yx : y \in H\}$ the right coset of the subgroup H. Define the relation $\sim$ on G of right congruence by $a \sim b$ if and only if $ab^{-1} \in H$. This relation is equivalence on G. Hence the right cosets of H are pairwise disjoint. The cardinal of the collection of all right cosets is called the index of the subgroup H in the group G and is denoted by $|G : H|$. Let $H \subset G$ is a subgroup with index $r \in N \cup \{+\infty\}$. Denote by $H_1 = H, H_2, ..., H_r$ the right cosets of H. For an arbitrary finite subset (not necessary subgroup) $A \subset G$ denote

$$q^H_{A,i} = |A \cap H_i|, \ i = 1, ..., r, \tag{1}$$

where $|S|$ denotes the cardinal of S.

Definition 2.1. *Let A be a subset of G and $\overrightarrow{k} = (k_1, ..., k_m)$ be a vector with*

$$k_1, ..., k_m \in \{1, ..., |A|\}, \quad k_1 \le k_2 \le ... \le k_m \quad \text{and} \quad \sum_{i=1}^{m} k_i = |A|. \tag{2}$$

A subgroup H of index $r \ge m$ is called $(A, \overrightarrow{k})$-ground subgroup if there are $i_1, ..., i_m \in \{1, ..., r\}$, $i_p \ne i_q$, $p \ne q$ such that $q^H_{A,i_j} = k_j, j = 1, .., m$.

Denote by $\mathcal{H}(A, \overrightarrow{k}) \equiv \mathcal{H}_G(A, \overrightarrow{k})$ the set of all $(A, \overrightarrow{k})-$ ground subgroups of a group G. For $\mathcal{H}(A, \overrightarrow{k}) \ne \oslash$ we denote

$$r_0 \equiv r_0(A, \overrightarrow{k}) = \min\{|G : H| : H \in \mathcal{H}(A, \overrightarrow{k})\}. \tag{3}$$

Remark 2.1. *The ground subgroups can be similarly defined for the left cosets of H.*

Remark 2.2. *If $A \subset G$ and a subgroup H of index $r \ge 1$ are given then it is obvious that the subgroup H is $(A, \overrightarrow{k})$-ground subgroup with $\overrightarrow{k} = (k_1, ..., k_m)$ such that $k_j = |A \cap H_j| \ne 0$, $j = 1, ..., m$. Thus $\mathcal{H}(A) = \bigcup_{\overrightarrow{k}} \mathcal{H}(A, \overrightarrow{k})$ is the set of all subgroups of G. Moreover if we know all subgroups of a group G then for a given $(A, \overrightarrow{k})$ it will be easy to describe the set $\mathcal{H}(A, \overrightarrow{k})$. But the inverse problem: for a given A and $\overrightarrow{k}$ with condition (2) finding of an $(A, \overrightarrow{k})$-ground subgroup is not easy.*

So the main problem of this paper is

Problem 1. Let G be a finite or countable group. For a given $A \subset G$ and $\overrightarrow{k}$ with conditions (2) describe $\mathcal{H}(A, \overrightarrow{k})$ and find $r_0 = r_0(A, \overrightarrow{k})$.

Remark 2.3. *Since $|A \cap H_i| = |gA \cap H_i|$ for any $g \in H$ and $i = 1, .., r$, we have that if H is an $(A, \overrightarrow{k})$-ground subgroup then it is $(gA, \overrightarrow{k})$- ground subgroup for any $g \in H$.*

3. Groups without proper subgroups

Recall that in every group the set containing only the identity element 1, and the group itself, are subgroups. Subgroups other than these are called proper subgroups. In this section we consider groups which have no proper subgroups. Such a group is finite and with a prime order.

One can easily prove the following

Proposition 3.1. *If G is a group without proper subgroups then for all $A \subset G$*

1. $\mathcal{H}(A, \overrightarrow{k}) = \{\{1\}\}$ *and* $r_0 = |G|$ *if* $\overrightarrow{k} = (k_1, ..., k_m)$ *with* $k_i = 1$, *for all* $i = 1, ..., m$.

2. $\mathcal{H}(A, \overrightarrow{k}) = \{G\}$ *and* $r_0 = 1$ *if* $\overrightarrow{k} = (|A|)$ *i.e.* $m = 1$.

3. $\mathcal{H}(A, \overrightarrow{k}) = \oslash$ *if* $\overrightarrow{k} = (k_1, ..., k_m)$ *with at least one* $k_i \in \{2, ..., |A| - 1\}$.

4. Cyclic groups

Let G be a cyclic group of order n (n may be infinity). It is known that every subgroup of G is cyclic. Moreover, the order of any subgroup of G is a divisor of n and for each positive divisor p of n the group G has exactly one subgroup of order p. Each infinite subgroup of G is pZ for some p, which is bijective to (so isomorphic to) Z. All factor groups of Z are finite, except for the trivial exception $Z/0 = Z/0Z$.

The following theorem gives upper estimation for the set $\mathcal{H}$.

Theorem 4.1. *Let G be a cyclic group of finite order n and $1 = n_0 < n_1 < ... < n_{p-1} < n_p = n$ are all divisors of n. Let $H^{(n_i)}$ be the subgroup of G with index $n_i, i = 0, ..., p$. For $A \subset G$ and $\overrightarrow{k}$ with conditions (2) we have*

$$\mathcal{H}(A, \overrightarrow{k}) \subseteq \widehat{\mathcal{H}}(A, \overrightarrow{k}) = \left\{ H^{(n_i)} : m \leq n_i \leq \frac{n}{\| \overrightarrow{k} \|} \right\},$$

where $\| \overrightarrow{k} \| = \max\{k_i : 1 \leq i \leq m\}$.

Proof. By construction of $\overrightarrow{k}$ we have $m \leq n_i$. Let $H_j^{(n_i)}, j = 1, 2, ..., n_i$ be the right cosets of $H^{(n_i)}$. We have

$$q_{A,j}^{H^{(n_i)}} = |A \cap H_j^{(n_i)}| = k_j \leq \frac{n}{n_i},$$

i.e. $n_i \leq \frac{n}{k_j}$ for all $j = 1, .., m$ which implies $n_i \leq \frac{n}{\| \overrightarrow{k} \|}$. This completes the proof. □

Corollary 4.1. *If $m > \frac{n}{\| \overrightarrow{k} \|}$ then $\mathcal{H}(A, \overrightarrow{k}) = \oslash$.*

The following example shows that the estimation of the set $\mathcal{H}(A, \overrightarrow{k})$ given in Theorem 4.1 can not be improved i.e there are some A and $\overrightarrow{k}$ such that $\mathcal{H}(A, \overrightarrow{k}) = \widehat{\mathcal{H}}(A, \overrightarrow{k})$ and even there are some A and $\overrightarrow{k}$ such that $\mathcal{H}(A, \overrightarrow{k}) = \oslash$.

Example 1. Consider the group $G = \{0, 1, 2, 3, 4, 5\}$ under addition modulo 6. All subgroups of G are

$$H^{(1)} = G, \ H^{(2)} = \{0, 2, 4\}, \ H^{(3)} = \{0, 3\}, \ H^{(6)} = \{0\}.$$

Consider $A = A_1 = \{1, 2, 3\}$ then $\overrightarrow{k}$ can be one of the following vectors:

$$\overrightarrow{k}^{(1)} = (1, 1, 1), \ \overrightarrow{k}^{(2)} = (1, 2), \ \overrightarrow{k}^{(3)} = (3)$$

and we have

$$\mathcal{H}(A, \overrightarrow{k}^{(1)}) = \left\{ H^{(3)}, H^{(6)} \right\} = \widehat{\mathcal{H}}(A, \overrightarrow{k}^{(1)}),$$

$$\mathcal{H}(A, \overrightarrow{k}^{(2)}) = \left\{ H^{(2)} \right\} \subset \widehat{\mathcal{H}}(A, \overrightarrow{k}^{(2)}), \mathcal{H}(A, \overrightarrow{k}^{(3)}) = \left\{ H^{(1)} \right\} \subset \widehat{\mathcal{H}}(A, \overrightarrow{k}^{(3)}).$$

Hence if $A = A_1$ for all $\overrightarrow{k}$ we have non-empty set of ground subgroups. Now consider the case $A = A_2 = \{0, 1, 2, 4\}$ then $\overrightarrow{k}$ can be one of the following vectors:

$$\overrightarrow{k}^{(1)} = (1, 1, 1, 1), \ \overrightarrow{k}^{(2)} = (1, 1, 2), \ \overrightarrow{k}^{(3)} = (1, 3), \ \overrightarrow{k}^{(4)} = (2, 2), \overrightarrow{k}^{(5)} = (4)$$

and we obtain

$$\mathcal{H}(A, \overrightarrow{k}^{(1)}) = \left\{H^{(6)}\right\} \subset \widehat{\mathcal{H}}(A, \overrightarrow{k}^{(1)}),$$

$$\mathcal{H}(A, \overrightarrow{k}^{(2)}) = \left\{H^{(3)}\right\} = \widehat{\mathcal{H}}(A, \overrightarrow{k}^{(2)}),\ \mathcal{H}(A, \overrightarrow{k}^{(3)}) = \left\{H^{(2)}\right\} = \widehat{\mathcal{H}}(A, \overrightarrow{k}^{(3)}),$$

$$\mathcal{H}(A, \overrightarrow{k}^{(4)}) = \emptyset,\ \mathcal{H}(A, \overrightarrow{k}^{(5)}) = \left\{H^{(1)}\right\} = \widehat{\mathcal{H}}(A, \overrightarrow{k}^{(5)}).$$

So for $A = A_2$ we can have all possible cases: empty set, subset and equality. Now we shall consider the case $n = \infty$ i.e $G = Z$. Let A be a finite subset of Z and $p \in N$. Denote

$$A_{p,i} = \{x \in A : x = i(\text{mod} p)\},\ i = 0, 1, ..., p - 1.$$

The following proposition is obvious

Proposition 4.1. *If $A \subset Z$ and $\overrightarrow{k} = (k_1, ..., k_m)$ with (2) are given, then $pZ \in \mathcal{H}(A, \overrightarrow{k})$ if and only if for any $j \in \{1, ..., m\}$ there is $i_j \in \{0, 1, ..., p-1\}$ such that $|A_{p,i_j}| = k_j$.*

Points $x, y \in Z$ are called nearest-neighbors if $|x - y| = 1$.

Proposition 4.2. *Let A be an arbitrary finite subset of Z. Assume that there exists $q \in \{2, ..., |A|\}$ such that any $B \subset A$ with $|B| = q$ has at least one pair of nearest-neighbors. Then $\mathcal{H}(A, \overrightarrow{k}) = \emptyset$ for any $\overrightarrow{k} = (k_1, ..., k_m)$ with $m > 1$ and at least one coordinate $k_i = q$.*

Proof. For $m > 1$, it is easy to see that $Z \notin \mathcal{H}(A, \overrightarrow{k})$. Assume that there is $p > 1$ such that $pZ \in \mathcal{H}(A, \overrightarrow{k})$ then by definition we should have $|A \cap (pZ + r)| = q$ for some $r \in \{0, 1, ..., p-1\}$. Denote $B = A \cap (pZ + r)$, by conditions of theorem we have $x, y \in B$ such that $x = y + 1$. Since $x, y \in pZ + r$, $x - y = 0(\text{mod} p)$ i.e. $1 = 0(\text{mod} p)$ and $p = 1$ this is contradiction to $p > 1$. □

5. Free groups

Let G be a countable free group. For $A \subset G$ we denote

$$A^* = \{z \in G : \exists x, y \in A, z = xy^{-1}\}. \tag{4}$$

If G is generated by a set M, then M is called an irreducible set of generators if no proper subset of M is a set of generators for G.

Lemma 5.1. *1. $1 \in A^*$, where 1 is the identity of G. Moreover $A^* = \{1\}$ iff $|A| = 1$; 2. $A = A^*$ iff A is a subgroup;*

3. $A \cap A^* = \emptyset$ iff A is a subset of an irreducible set of generators of G;

4) $A \subset A^*$ iff $1 \in A$.

Proof. 1)-3) are straightforward.

4) If $1 \in A$ then for any $x \in A$ we have $x1^{-1} = x \in A^*$ i.e $A \subset A^*$. Now assume $A \subset A^*$ and $1 \notin A$ then since A is a set as $A = \{x_1, ..., x_q\}$ we should have $1 \neq x_m = x_i x_j^{-1}$ for any $m = 1, ..., q$ and some $i = i(m), j = j(m) \in \{1, ..., q\}, i \neq j$ i.e. $x_m x_j x_i^{-1} = 1$ this is contradiction to the assumption that G is a free group. This completes the proof. □

Note that if $\mathbf{1} = (1, ..., 1)$ is the vector defined by conditions (2) then for an arbitrary $A \subset G$ we have $\{1\} \in \mathcal{H}(A, \mathbf{1})$. But $|G : \{1\}| = \infty$. Now we shall give a construction of an $(A, \mathbf{1})$-ground subgroup with finite index.

We shall use the following

Theorem 5.1. [6] *If $x \in G \setminus \{1\}$, then there exists a normal subgroup H_x of G, of finite index, that does not contain x.*

Denote

$$H_A = \bigcap_{x \in A^* \setminus \{1\}} H_x, \tag{5}$$

where $A \subset G$ with $|A| > 1$, A^* is defined by (4) and H_x is given in Theorem 5.1.

Denote by $\alpha(x)$ the number of these generators that occur in the reduced form of $x \in G$ and by $|x|$ the length of x.

Proposition 5.1. *The normal subgroup $H_A \subset G$ is a $(A, \mathbf{1})$-ground subgroup for any $A \subset G$, with $|A| > 1$. Moreover*

$$\min_{x \in A^* \setminus \{1\}} (\alpha(x) + 1) \leq |G : H_A| \leq \prod_{x \in A^* \setminus \{1\}} (|x| + 1)!. \tag{6}$$

Proof. Let $H_{A,j}$ be a right coset of H_A. By our construction of H_A we get $H_A \cap A^* = \{1\}$, consequently, for any $x, y \in A$ we have $xy^{-1} \notin H_A$, i.e $|A \cap H_{A,j}| \in \{0, 1\}$, for any j. Following the proof of Theorem 5.1, given in [6] page 42, one can easily see that

$$\alpha(x) + 1 \leq |G : H_x| \leq (|x| + 1)!, \tag{7}$$

which by (5) gives (6).

$\square$

6. An application: a model on a Cayley tree

In this section we consider a model of statistical mechanics on a Cayley tree. The Cayley tree $\Gamma^k = (V, L)$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k + 1$ edges issue. Here V is the set of vertices and L is the set of edges of Γ^k.

It is known that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k + 1$ cyclic groups $\{e, a_i\}$, $i = 1, ..., k + 1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators $a_1, a_2, ..., a_{k+1}$.

For $A \subseteq V$ a spin *configuration* σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi = \{1, 2, ..., q\}$; the set of all configurations coincides with $\Omega_A = \Phi^A$. We denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$. Define a *periodic configuration* as a configuration $\sigma \in \Omega$ which is invariant under a subgroup of shifts $G_k^* \subset G_k$ of finite index.

In [10] the following model of statistical mechanics was considered. For $A \subset V$ define the function $U : \sigma_A \in \Omega_A \to U(\sigma_A) \in \big\{|A| - 1, |A| - 2, ..., |A| - \min\{|A|, |\Phi|\}\big\}$ by

$$U(\sigma_A) = |A| - |\sigma_A \cap \Phi|, \tag{8}$$

where $\Phi = \{1, 2, ..., q\}$ and $|\sigma_A \cap \Phi|$ is the number of distinct values of $\sigma_A(x), x \in A$. For instance if σ_A is a constant configuration then $|\sigma_A \cap \Phi| = 1$.

Note that if $|A| = 2$, say $A = \{x, y\}$, then $U(\{\sigma(x), \sigma(y)\}) = \delta_{\sigma(x),\sigma(y)}$ i.e U is a generalization of the Kronecker symbol.

Denote by M_r the set of all balls $b_r(x) = \{y \in V : d(x, y) \leq r\}$ with radius $r \geq 1$, where $d(x, y)$ is the distance on the Cayley tree i.e the number of edges of the shortest path connecting x and y.

Now consider the Hamiltonian

$$\mathbf{H}(\sigma) = -J \sum_{b \in M_r} U(\sigma_b), \tag{9}$$

where $J \in R \setminus \{0\}$.

Remark 6.1. *Since U is a generalization of the Kronecker symbol, the Hamiltonian (9) is a natural generalization of Potts model, for details about Potts model see e.g.* [2], [11].

A periodic configuration $\phi \in \Omega$ is called a *ground state* of the Hamiltonian (9) if $\mathbf{H}(\phi) \leq \mathbf{H}(\sigma)$ for all $\sigma \in \Omega$.

Denote by $GS(\mathbf{H})$ the set of all ground states of $\mathbf{H}$. Put

$$K \equiv K(k,r) = |b_r(x)| = 1 + (k-1)^{-1}(k+1)(k^r - 1),$$

where $k \geq 2$ is the order of the Cayley tree and $r \geq 1$ is an integer number.

Theorem 6.1. *1. If $J > 0$ then $GS(\mathbf{H})$ contains the constant configurations only i.e it contains q configurations $\sigma^{(i)} \equiv i$, $i = 1, .., q$.*

2. If q is large enough and $J < 0$ then $GS(\mathbf{H})$ contains at least $\frac{q!}{(q-K)!}$ periodic ground states, which are periodic with respect to the $(b, \mathbf{1})$-ground subgroup H_b, with $b \in M_r$.

Proof. 1) It is easy to see that for $J > 0$ a configuration ϕ minimizes $\mathbf{H}$ iff it maximizes U on any ball $b \in M_r$ i.e iff it is a constant configuration. 2) For $J < 0$ a configuration ϕ minimizes $\mathbf{H}$ iff it also minimizes U on any ball $b \in M_r$. So on any ball $b \in M_r$ the configuration has to be with distinct values. First consider $b(e) \in M_r$ i.e the ball with the center e. Note that an analogue of Theorem 5.1 is true for the group G_k (see [2]). Consequently, Proposition 5.1 is also true for G_k. Hence we have a $(b(e), \mathbf{1})$-ground subgroup H of G_k. It is easy to see that this subgroup is also $(b(g), \mathbf{1})$ -ground subgroup for any $g \in G_k$, where $b(g) \in M_r$ is a ball with the center g. So now we can define a H-periodic ground state ϕ as a function $\phi : G_k \to \Phi$ such that $\phi(x) = i$ if x is an element of the (right) coset H_i of H. Since q is large enough we can always choose distinct values of the configuration ϕ on distinct cosets and number of such choices is equal to $\binom{q}{K} \cdot K! = \frac{q!}{(q-K)!}$. Theorem is proved. $\square$

Remark 6.2. *By the proof 2) of Theorem 6.1 it is clear that if we know $|G_k : H|$ then q can be chosen as $q \geq |G_k : H|$.*

7. Discussion and open problems

For a given finite or countable group G we have defined a concept of ground subgroup for a subset A, and its given partition. If groups are finite, we obtained a set of such subgroups and showed that this set can be empty for some suitable choice of A and its partition. For free groups we proved that for an arbitrary A one can construct a subgroup which divides the set A to distinct (non-equivalent, see section 2) elements. In the section 6 we applied this result to describe the ground states of a model with an arbitrary finite interaction radius (i.e diameter of the balls: $2r$). One of the key problems related to the (spin) models is the description of the set of Gibbs measures. This problem has a good connection with the problem of the description the set of ground states. Because the phase diagram of Gibbs measures is close to the phase diagram of the ground states for sufficiently small temperatures (see e.g.[3], [11]).

Results of the paper show that, in general, the Problem 1 (see section 2) is very difficult. For example, I do not have any deep result about exact value of r_0 (see Problem 1).

Note that from Problem 1 one can get more simple problem: let $A \subset G$ and its partition A_i, $i = 1, ..., m$, with $|A_i| = k_i$ are given, if a subgroup H is $(A, \overrightarrow{k})$ -ground subgroup then we can assume that $A_1 \subset H$ but $A_i \cap H = \emptyset$ for any $i = 2, ..., m$. So if we denote $B = A \setminus A_1$ then the following problem is a particular case of our problem:

Problem 2. Let G is a finite or countable group. Consider arbitrary subsets $A, B \in G$ such that $1 \notin B$ and $A \cap B = \emptyset$. Describe the set $\mathcal{H}(A, B)$ of all subgroups $H = H(A, B)$ of G such that $A \subset H$ and $B \cap H = \emptyset$?

It is easy to see that $\mathcal{H}(A, B)$ can be empty set. This happens, for example, if $A = \{a, b\}$ and $B = \{ab\}$ then from $A \subset H$ it follows $B \subset H$.

Note that for an arbitrary $A \subset G$ one can easily show that there is H with $A \subset H$. In section 5, for free groups we have showed that for any $B \subset G$ there exists a subgroup H_B such that $B \cap H_B = \emptyset$ if $1 \notin B$. Even for free groups I have not any idea how to construct a subgroup (if exists) with the properties mentioned in the Problem 2.

The following problem is very general:

Problem 3. Let $A_1, A_2, ..., A_m \subset G$ such that $A_i \cap A_j = \emptyset$, $i \neq j$. Is there any subgroup H of G, with index $\geq m$, such that A_i is a subset of a right coset $H_{p(i)}$ and $p(i) \neq p(j)$ for $i \neq j$?

Note that Problem 1 is a particular case of the Problem 3. Indeed any subgroup H satisfying the conditions of the Problem 3 also satisfies the conditions of the Problem 1, with $A = \bigcup_i A_i$.

8. Acknowledgements

I thank the Department of Algebra, University of Santiago de Compostela, Spain, for providing financial support of my visits to the Department (2010-2011). I thank the Abdus Salam International Center for Theoretical Physics (ICTP), Trieste, Italy for providing financial support of my several visits to ICTP.

References

[1] Cohen, D.E., Lyndon, R.C., (1963), Free bases for normal subgroups of free groups, Trans. Amer. Math. Soc. 108, pp.526-537.

[2] Ganikhodjaev, N.N., Rozikov, U.A., (1997), Description of periodic extreme Gibbs measures of some lattice model on the Cayley tree, Theor. Math. Phys., 111, pp.480-486.

[3] Georgii, H-O., (1988), Gibbs Measures and Phase Transitions, Walter de Gruyter, Berlin, 525p.

[4] Kargapolov, M.I., Merzljakov, Ju.I., (1979), Fundamentals of the theory of groups, Springer-Verlag New York.

[5] Keller, T.M., (2006), Fixed conjugacy classes of normal subgroups and the $k(GV)$-problem. J. Algebra, 305, pp.457-486.

[6] Kurosh, A.G., (1960), The Theory of Groups, II, Chelsea Publ. Com. New York.

[7] Normatov, E.P., Rozikov, U.A., (2006), Description of harmonic functions using group representations of the Cayley tree. Math. Notes., 79, pp.434-444.

[8] Rozikov, U.A., (1997), Partition structures of the Cayley tree and applications for describing periodic Gibbs distributions. Theor. Math. Phys., 112, pp.929-933.

[9] Rozikov, U.A., (2002), Representation of trees and its applications. Math. Notes., 72, pp.516-527.

[10] Rozikov U.A., (2008), A contour method on Cayley trees. Jour. Stat. Phys., 130, pp.801-813.

[11] Sinai, Ya.G., (1982), Theory of Phase Transitions: Rigorous Results, Pergamon, Oxford.

[12] Young, J.W., (1927), On the partitions of a group and the resulting classification. Bull. Amer. Math. Soc., 33, pp.453-461.

Utkir Rozikov graduated from Samarkand State University in 1993. He got Ph.D. degree in 1995 and Doctor of Sciences degree in physics and mathematics in 2001 at the Institute of Mathematics (Tashkent, Uzbekistan). He is known for his works on the theory of Gibbs measures of models on trees of statistical mechanics. He developed a contour method to study the models on trees and described complete set of periodic Gibbs measures. He has (with F. Mukhamedov) an important contribution to non-Archimedean theory of phase transitions and dynamical systems.

He has important results on non-Volterra quadratic operators. He gave with N. Ganikhodjaev a construction of a quadratic operator which connects the phases of the statistical mechanics models with genetics models. The result allows to use the thermodynamics in physical biology. He was invited to several leading universities and research centers in the UK, France, Italy and Germany etc. His excellent papers (about 70) are published in the leading journals.